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Anti fuzzy B-subalgebras under S-norms

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Abstract

In this paper, we apply the concept of s-norm S to fuzzy structure of B-algebras. The notion of an anti fuzzy B-subalgebra and an anti fuzzy normal B-subalgebra with respect to s-norm are introduced and several related properties are investigated. The union and direct sum of them are defined and investigated. Finally, by using B-homomorphisms of B-algebras, characterizations of them are given.

Keywords: Algebra and orders, theory of fuzzy sets, norms, sums and unions, homomorphisms.

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1. Introduction

Neggers and Kim [7, 8] introduced a new notion, called a B-algebras which is related to several classes of algebras. The concept of a fuzzy set, which was introduced by Zadeh in his definitive paper [19] of 1965, was applied by many researchers to generalize some of the basic concepts of algebras. The fuzzy algebraic structures play a vital role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, real analysis, measure theory etc. In 2002, Jun et al. [5] applied the concept of fuzzy sets to B-algebras. In 2003, Ahn and Bang [2] discussed some results on fuzzy subalgebras in B-algebras. Saeid introduced the notion of fuzzy topological B-algebras [17]. Norms are operations which generalize the logical disjunction to fuzzy logic. Senapati et al. [18] investigated fuzzy subalgebras of B-algebras under t-norms. In previous works [9, 10, 11, 12, 13, 14, 15, 16], by using norms, we investigated some properties of fuzzy algebraic structures. In this paper, we introduce the conceps of anti fuzzy B-subalgebras and anti fuzzy normal B-subalgebras with respect to s-norms and discuss the level B-subalgebras. Next we state and prove some theorems which determine the relationship between these notions and B-subalgebras and we consider characterizations of them. Also we define union and direct sum of them and investigate related topics. Finally, we define images and inverse images of them and we study how the homomorphic images and inverse images of them under B-homomorphisms of B-algebras become anti fuzzy B-subalgebras and anti fuzzy normal B-subalgebras with respect to s-norms, respectively.

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2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal.

Definition 2.1. (See [5]) A B-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:
(1) $x * x = 0$,
(2) $x * 0 = x$,
(3) $(x * y) * z = x * (z * (0 * y))$
for all $x, y, z \in X$. A partial ordering $\leqslant$ on X can be defined by $x \leqslant y$ if and only if $x * y = 0$. A non-empty subset N of a B-algebra X is called a B-subalgebra of X if $x * y \in N$ for any $x, y \in N$. A non-empty subset N of a B-algebra X is said to be normal if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B-algebra X is a B-subalgebra of X, but the converse need not be true. A non-empty subset N of a B-algebra X is called a normal B-subalgebra of X if it is both a B-subalgebra and normal

Example 2.2. (See [5]) Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y = \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra.

Example 2.3. (See [5]) Let $\mathbb{Z}$ be the group of integers under usual addition and let $\alpha \notin \mathbb{Z}$. We adjoin the special element α to $\mathbb{Z}$. Let $X = \mathbb{Z} \cup \alpha$. Define $\alpha + 0 = \alpha$, $\alpha + n = n - 1$ where $n \neq 0$ in $\mathbb{Z}$ and $i\alpha + \alpha$ an arbitrary element in X. Define a mapping $\varphi : X \to X$ by $\varphi(\alpha) = 1, \varphi(n) = -n$ where $n \in \mathbb{Z}$. If we define a binary operation $*$ on X by $x * y = x + \varphi(y)$, then $(X; *, 0)$ is a non-group derived B-algebra.

Lemma 2.4. (See [5]) If X is a B-algebra, then $x = 0 * (0 * x)$ for all $x \in X$.

Definition 2.5. (See [3]) A mapping $f : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ of B-algebras is called a B-homomorphism if $f(x * y) = f(x) \acute{*} f(y)$, for all $x, y \in X$. The zero mapping $\theta : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ of B-algebras with $\theta(0) = \acute{0}$ is a B-homomorphism.

Definition 2.6. (See [4]) Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \leqslant t\}$ is called a lower level of μ.

Definition 2.7. (See [9]) Let φ be a function from set X into set Y such that $\mu : X \to [0, 1]$ and $\nu : Y \to [0, 1]$. For all $x \in X, y \in Y$, we define $\varphi(\mu)(y) = \inf\{\mu(x) \mid x \in X, \varphi(x) = y\}$ and $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$.

Definition 2.8. (See [4]) An s-norm S is a function $S : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(1) $S(x, 0) = x$,
(2) $S(x, y) \leqslant S(x, z)$ if $y \leqslant z$,
(3) $S(x, y) = S(y, x)$,
(4) $S(x, S(y, z)) = S(S(x, y), z)$,
for all $x, y, z \in [0, 1]$.

We say that S is idempotent if for all $x \in [0, 1], S(x, x) = x$.

Example 2.9. (See [4]) The basic s-norms are $S_m(x, y) = \max\{x, y\}$, $S_b(x, y) = \min\{1, x + y\}$ and $S_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$.
S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

Definition 2.10. (See [6]) The function $S_n : \prod_{i=1}[0,1] \to [0,1]$ is defined by

$$S_n(x_1, x_2, ..., x_n) = S(x_i, S_{n-1}(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n))$$

for all $1 \leqslant i \leqslant n$, where $n \geqslant 2$ such that $S_2 = S$ and $S_1 = id$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 2.11. (See [6]) For every s-norm S and every $x_i, y_i \in [0,1]$, where $1 \leqslant i \leqslant n$, and $n \geqslant 2$, we have

$$S_n(S(x_1, y_1), S(x_2, y_2), ..., S(x_n, y_n)) = S(S_n(x_1, x_2, ..., x_n), S_n(y_1, y_2, ..., y_n)).$$

Lemma 2.12. (See [6]) For a s-norm S and every $x_1, x_2, ..., x_n \in [0,1]$, where $n \geqslant 2$, we have

$$S_n(x_1, x_2, ..., x_n) = S(...S(S(S(x_1, x_2), x_3), x_4), x_n) = S(x_1, S(x_2, S(x_3, ...S(x_{n-1}, x_n)...)))$$

Definition 2.13. (See [9]) Let $\mu, \nu : X \to [0,1]$ and S be a s-norm. We define the intersection of μ and ν as

$$\mu \cup \nu : X \to [0,1]$$

by

$$(\mu \cup \nu)(x) = S(\mu(x), \nu(x))$$

for all $x \in X$.
Let $\mu : X \to [0,1]$ and $\nu : Y \to [0,1]$ and S be a s-norm. The direct sum of μ and ν is denoted by

$$\mu \oplus \nu : X \oplus Y \to [0,1]$$

is defined by

$$(\mu \oplus \nu)(x, y) = S(\mu(x), \nu(y))$$

for all $(x, y) \in X \oplus Y$.

Lemma 2.14. (See [1]) Let S be a s-norm. Then

$$S(S(x, y), S(w, z)) = S(S(x, w), S(y, z))$$

for all $x, y, w, z \in [0,1]$.

3. Anti fuzzy B-algebras under s-norms

Definition 3.1. Let X be a B-algebra. Define $\mu : X \to [0,1]$ an anti fuzzy B-algebra under t-norm T if it satisfies the following inequalities:

$$\mu(x * y) \leqslant S(\mu(x), \mu(y))$$

for all $x, y \in X$.
Denote by $AFBS(X)$, the set of all anti fuzzy B-algebras of B-algebra X under s-norm S.

Example 3.2. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.35 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.55 & \text{if } x = 2, \end{cases}$$

Let $S(a, b) = S_p(a, b) = a + b - ab$ for all $a, b \in [0,1]$ then $\mu \in AFBS(X)$.

Proposition 3.3. Let $\mu \in AFBS(X)$ and S be idempotent. Then $\mu(0) \leqslant \mu(x)$ for all $x \in X$.

Proof. Let $x \in X$ and then $\mu(0) = \mu(x * x) \leqslant S(\mu(x), \mu(x)) = \mu(x)$. □

Proposition 3.4. Let $\mu \in AFBS(X)$. Then
(1) $\mu(0 * x) = \mu(x)$,
(2) $\mu(x * (0 * y)) \leqslant S(\mu(x), \mu(y))$,
for all $x, y \in X$.

Proof. Let $x, y \in X$. Then
(1)
$$\mu(0 * x) \leqslant S(\mu(0), \mu(x)) \leqslant S(\mu(x), \mu(x)) = \mu(x) = \mu(0 * (0 * x)) \leqslant S(\mu(0), \mu(0 * x)) = \mu(0 * x)$$
thus $\mu(0 * x) = \mu(x)$.
(2)
$$\mu(x * (0 * y)) \leqslant S(\mu(x), \mu(0 * y)) = S(\mu(x), \mu(y)).$$
□

Proposition 3.5. If a fuzzy set μ in X satisfies (1) and (2) in Proposition 3.4, then $\mu \in AFBS(X)$.

Proof. Let $x, y \in X$. Then
$$\mu(x * y) = \mu(x * (0 * (0 * y))) \leqslant S(\mu(x), \mu(0 * (0 * y))) = S(\mu(x), \mu(0 * y)) = S(\mu(x), \mu(y))$$
so $\mu(x * y) \leqslant S(\mu(x), \mu(y))$ then $\mu \in AFBS(X)$. □

Proposition 3.6. Let $\mu : X \to [0, 1]$ be a fuzzy set and S be idempotent s-norm. If $\mu \in AFBS(X)$, then the set
$$\mu_t = \{x \in X : \mu(x) \leqslant t\}$$
is either empty or subalgebra of B-algebra X for every $t \in [0, 1]$.

Proof. Let $\mu \in AFBS(X)$ and $\mu_t = \{x \in X : \mu(x) \leqslant t\}$ be not empty and $x, y \in \mu_t$. Then $\mu(x) \leqslant t$ and $\mu(y) \leqslant t$ such that
$$\mu(x * y) \leqslant S(\mu(x), \mu(y)) \leqslant S(t, t) = t$$
which means that $x * y \in \mu_t$ and so the set μ_t will be subalgebra of B-algebra X for every $t \in [0, 1]$. □

Example 3.7. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set given by the following Cayley table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define
$$\mu : X \to [0, 1]$$
as
$$\mu(x) = \begin{cases} 0.16 & \text{if } x = 0, \\ 0.25 & \text{if } x = 1, \\ 0.34 & \text{if } x = 2, \\ 0.43 & \text{if } x = 3, \\ 0.52 & \text{if } x = 4, \\ 0.61 & \text{if } x = 5, \end{cases}$$

Let $S(a,b) = S_p(a,b) = a + b - ab$ for all $a, b \in [0,1]$ then $\mu \in AFBS(X)$.
Let $t = 0.4$ then

$$\mu_{0.4} = \{x \in X : \mu(x) \leqslant 0.4\} = \{0, 1, 2\}$$

with the following Cayley table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

so $\mu_{0.4}$ will be subalgebra of B-algebra X.

Proposition 3.8. Any subalgebra of a B-algebra X can be realized as a level subalgebra of some $AFBS(X)$.

Proof. Let A be a subalgebra of a given B-algebra X and let $\mu : X \to [0,1]$ be a fuzzy subalgebra defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 1, & \text{if } x \notin A, \end{cases}$$

such that $t \in (0,1)$ is fixed. It is clear that $\mu_t = A$. We must prove that $\mu \in AFBS(X)$.
If $x, y \in A$, then $x * y \in A$ and hence $\mu(x) = \mu(y) = \mu(x * y) = t$ which means that

$$\mu(x * y) = t \leqslant t = S(t,t) = S(\mu(x), \mu(y))$$

and so $\mu \in AFBS(X)$.
If If $x, y \notin A$, then $\mu(x) = \mu(y) = 1$ which implies that

$$\mu(x * y) \leqslant 1 = S(1,1) = S(\mu(x), \mu(y))$$

and then $\mu \in AFBS(X)$.
If $x \in A$ and $y \notin A$, then $\mu(x) = t < 1 = \mu(y)$ therefore

$$\mu(x * y) = 1 \leqslant 1 = \mu(y) = S(\mu(x), \mu(y))$$

so $\mu \in AFBS(X)$. □

Proposition 3.9. Let $\mu_1, \mu_2 \in\in AFBS(X)$. Then $\mu_1 \cup \mu_2 \in\in AFBS(X)$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned}(\mu_1 \cup \mu_2)(x * y) &= S(\mu_1(x * y), \mu_2(x * y)) \\ &\leqslant S(S(\mu_1(x), \mu_1(y)), S(\mu_2(x), \mu_2(y))) \\ &= S(S(\mu_1(x), \mu_2(x)), S(\mu_1(y), \mu_2(y))) \\ &= S((\mu_1 \cup \mu_2)(x), (\mu_1 \cup \mu_2)(y))\end{aligned}$$

which means that $\mu_1 \cup \mu_2 \in AFBS(X)$. □

Example 3.10. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

Then $(X, *, 0)$ is a B-algebra. Define

$$\mu : X \to [0,1]$$

as

$$\mu(x) = \begin{cases} 0.15 & \text{if } x = 0, \\ 0.25 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.45 & \text{if } x = 3, \end{cases}$$

and

$$\nu : X \to [0,1]$$

as

$$\nu(x) = \begin{cases} 0.1 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.55 & \text{if } x = 3, \end{cases}$$

Let $S(a, b) = S_m(a, b) = \max\{a, b\}$ for all $a, b \in [0, 1]$. Then $\mu, \nu \in AFBS(X)$. Also

$$\mu \cup \nu : X \to [0,1]$$

as

$$(\mu \cup \nu)(x) = S(\mu(x), \nu(x)) = S_m(\mu(x), \nu(x)) = \max\{\mu(x), \nu(x)\} = \begin{cases} 0.15 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.55 & \text{if } x = 3, \end{cases}$$

so $\mu \cup \nu \in AFBS(X)$.

Corollary 3.11. Let $\mu_i \subseteq AFBS(X)$ for $i = 1, 2, 3, 4, ..., n$. Then $\cup_{i=1,2,3,...,n}\mu_i \in AFBS(X)$.

Proposition 3.12. Let $\mu \in AFBS(X)$ and S be idempotent s-norm. Then

$$A = \{x \in X : \mu(x) = \mu(0)\}$$

will be a subalgebra of X.

Proof. Let $x, y \in A$ then $\mu(x) = \mu(0) = \mu(y)$. As $\mu \in AFBS(X)$ so

$$\mu(x * y) \leqslant S(\mu(x), \mu(y)) = S(\mu(0), \mu(0)) = \mu(0) \leqslant \mu(x * y)$$

thus $\mu(x * y) = \mu(0)$ and then $x * y \in A$ thus A will be a subalgebra of X. □

As is well known, the anti characteristic function of a set is a special fuzzy set. Suppose A is a non-empty subset of X. By χ_A we denote the anti characteristic function of A , that is,

$$\chi_A(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \notin A. \end{cases}$$

Proposition 3.13. Let S be idempotent s-norm. Then A is a B-subalgebra of X if and only if $\chi_A \in AFBS(X)$.

Proof. Let $x, y \in X$.
(1) If $x, y \in A$, then $x * y \in A$ and then $\chi_A(x) = \chi_A(y) = \chi_A(x * y) = 0$. Thus

$$\chi_A(x * y) = 0 \leqslant 0 = S(0,0) = S(\chi_A(x), \chi_A(y)).$$

(2) If $x, y \notin A$, then $\chi_A(x) = \chi_A(y) = 1$. Thus

$$\chi_A(x * y) \leqslant 1 = S(1,1) = S(\chi_A(x), \chi_A(y)).$$

(3) If $x \in A$ and $y \notin A$, then $\chi_A(x) = 0$ and $\chi_A(y) = 1$. Then

$$\chi_A(x * y) \leqslant 1 = S(0,1) = S(\chi_A(x), \chi_A(y)).$$

(4) If $x \notin A$ and $y \in A$, then $\chi_A(x) = 1$ and $\chi_A(y) = 0$. So

$$\chi_A(x * y) \leqslant 1 = S(1,0) = S(\chi_A(x), \chi_A(y)).$$

Therefore (1)-(4) give us that $\chi_A \in AFBS(X)$.
Conversely, let $\chi_A \in AFBS(X)$ and $x, y \in A$. Then $\chi_A(x) = \chi_A(y) = 0$ Thus

$$\chi_A(x * y) \leqslant S(\chi_A(x), \chi_A(y)) = S(0,0) = 0$$

so $\chi_A(x * y) = 0$ which means that $x * y \in A$ and thus A will be a B-subalgebra of X. □

Proposition 3.14. Let $\mu \in AFBS(X)$ and $\nu \in AFBS(Y)$. Then $\mu \oplus \nu \in AFBS(X \oplus Y)$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \oplus Y$. Then

$$\begin{aligned}
(\mu \oplus \nu)((x_1, y_1) * (x_2, y_2)) &= (\mu \oplus \nu)(x_1 * x_2, y_1 * y_2) \\
&= S(\mu(x_1 * x_2), \nu(y_1 * y_2)) \\
&\leqslant S(S(\mu(x_1), \mu(x_2)), S(\nu(y_1), \nu(y_2))) \\
&= S(S(\mu(x_1), \nu(y_1)), S(\mu(x_2), \nu(y_2))) \\
&= S((\mu \oplus \nu)(x_1, y_1), (\mu \oplus \nu)(x_2, y_2))
\end{aligned}$$

thus

$$(\mu \oplus \nu)((x_1, y_1) * (x_2, y_2)) \leqslant S((\mu \oplus \nu)(x_1, y_1), (\mu \oplus \nu)(x_2, y_2))$$

and so $\mu \oplus \nu \in AFBS(X \oplus Y)$. □

Example 3.15. Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

and $Y = \{0, 1, 2\}$ be a set given by the following Cayley table:

*	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

Then $(X, *, 0)$ and $(Y, *, 0)$ will be two B-algebras. Define

$$\mu : X \to [0,1]$$

as

$$\mu(x) = \begin{cases} 0.15 & \text{if } x = 0, \\ 0.25 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \end{cases}$$

and

$$\nu : Y \to [0,1]$$

as

$$\nu(y) = \begin{cases} 0.1 & \text{if } y = 0, \\ 0.2 & \text{if } y = 1, \\ 0.3 & \text{if } y = 2. \end{cases}$$

Let $S(a,b) = S_b(a,b) = \min\{1, a+b\}$ for all $a, b \in [0,1]$. Then $\mu \in AFBS(X)$ and $\nu \in AFBS(Y)$. Now

$$\mu \oplus \nu : X \oplus Y = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\} \to [0,1]$$

as

$$\begin{aligned} (\mu \oplus \nu)(x,y) &= S(\mu(x), \nu(y)) \\ &= \min\{1, \mu(x) + \nu(y)\} \\ &= \begin{cases} 0.25 & \text{if } (x,y) = (0,0) \\ 0.35 & \text{if } (x,y) = (0,1) \\ 0.45 & \text{if } (x,y) = (0,2) \\ 0.35 & \text{if } (x,y) = (1,0) \\ 0.45 & \text{if } (x,y) = (1,1) \\ 0.55 & \text{if } (x,y) = (1,2) \\ 0.45 & \text{if } (x,y) = (2,0) \\ 0.45 & \text{if } (x,y) = (2,1) \\ 0.65 & \text{if } (x,y) = (2,2) \end{cases} \end{aligned}$$

thus $\mu \oplus \nu \in AFBS(X \oplus Y)$.

Proposition 3.16. Let $\mu_i \in AFBS(X_i)$, where $1 \leqslant i \leqslant n$, then

$$\mu = \oplus_{i=1}^{n} \mu_i \in AFBS(\oplus_{i=1}^{n} X_i = X)$$

such that

$$\mu(x) = (\oplus_{i=1}^{n} \mu_i)(x_1, x_2, ..., x_n) = S_n(\mu_1(x_1, \mu_2(x_2, ..., \mu_n(x_n))$$

for all $x = (x_1, x_2, ..., x_n) \in X$.

Proposition 3.17. If $\mu \in AFBS(X)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be an epimorphic B-homomorphism of B-algebras, then $\varphi(\mu) \in AFBS(Y)$.

Proof. Let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2$. Then

$$\begin{aligned} \varphi(\mu)(y_1 \acute{*} y_2) &= \inf\{\mu(x_1 * x_2) \mid x_1 * x_2 \in X, \varphi(x_1 * x_2) = y_1 \acute{*} y_2\} \\ &\leqslant \inf\{S(\mu(x_1), \mu(x_2)) \mid x_i \in X, \varphi(x_i) = y_i\} \\ &= S(\inf\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \inf\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\}) \\ &= S(\varphi(\mu)(y_1), \varphi(\mu)(y_2)) \end{aligned}$$

thus

$$\varphi(\mu)(y_1 \acute{*} y_2) \leqslant S(\varphi(\mu)(y_1), \varphi(\mu)(y_2)).$$

Thus $\varphi(\mu) \in AFBS(Y)$. □

Proposition 3.18. If $\nu \in AFBS(Y)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be a B-homomorphism of B-algebras, then $\varphi^{-1}(\nu) \in AFBS(X)$.

Proof. Let $x_1, x_2 \in X$.

$$\begin{aligned} \varphi^{-1}(\nu)(x_1 * x_2)) &= \nu(\varphi(x_1 * x_2)) \\ &= \nu(\varphi(x_1) * \varphi(x_2)) \\ &\leqslant S(\nu(\varphi(x_1), \nu(\varphi(x_2)) \\ &= S(\varphi^{-1}(\nu)(x_1)), \varphi^{-1}(\nu)(x_2)) \end{aligned}$$

then

$$\varphi^{-1}(\nu)(x_1 * x_2)) \leqslant S(\varphi^{-1}(\nu)(x_1)), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in AFBS(X)$. □

Example 3.19. Let $X = \{0, 1, 2, 3, 4, 5\}$ and $Y = \{0, 1, 2, 3\}$ be two sets given by the following Cayley tables:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

and

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

then $(X, *, 0)$ and $(Y, *, 0)$ will be two B-algebras. Define $\mu : X \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.15 & \text{if } x = 0, 1, \\ 0.25 & \text{if } x = 2, 3, \\ 0.35 & \text{if } x = 4, \\ 0.65 & \text{if } x = 5. \end{cases}$$

and $\nu : Y \to [0, 1]$ as

$$\nu(y) = \begin{cases} 0.25 & \text{if } y = 0, \\ 0.45 & \text{if } y = 1, 2, \\ 0.35 & \text{if } y = 3, \end{cases}$$

Let $S(a,b) = S_p(a,b) = a + b - ab$ for all $a, b \in [0,1]$ then $\mu \in AFBS(X)$ and $\nu \in AFBS(Y)$. Define B-homomorphism $\varphi : X \to Y$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, 1, \\ 1 & \text{if } x = 2, \\ 2 & \text{if } x = 3, \\ 3 & \text{if } x = 4, 5, \end{cases}$$

then we get that $\varphi(\mu) : Y \to [0,1]$ as

$$\varphi(\mu)(y) = \inf\{\mu(x) \mid x \in X, \varphi(x) = y\} = \begin{cases} 0.15 & \text{if } y = 0, \\ 0.25 & \text{if } y = 1, 2, \\ 0.35 & \text{if } y = 3, \end{cases}$$

and thus $\varphi(\mu) \in AFBS(Y)$. Also we will have that $\varphi^{-1}(\nu) : X \to [0,1]$ as

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) = \begin{cases} 0.25 & \text{if } x = 0, 1, \\ 0.45 & \text{if } x = 2, 3, \\ 0.35 & \text{if } x = 4, 5, \end{cases}$$

therefore $\varphi^{-1}(\nu) \in AFBS(X)$.

4. Anti fuzzy normal B-algebras under s-norms

Definition 4.1. Let X be a B-algebra. Define $\mu : X \to [0,1]$ an anti fuzzy normal B-algebra under s-norm S if it satisfies the following inequalities:

$$\mu((x * a) * (y * b)) \leqslant S(\mu(x * y), \mu(a * b))$$

for all $x, y, a, b \in X$.
Denote by $AFNBS(X)$, the set of all anti fuzzy normal B-algebras of B-algebra X under s-norm S.

Example 4.2. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.4 & \text{if } x = 0, \\ 0.5 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.7 & \text{if } x = 3. \end{cases}$$

Let $S(a,b) = S_b(a,b) = \min\{1, a + b\}$ for all $a, b \in [0,1]$ then $\mu \in AFNBS(X)$.

Now we prove that every anti fuzzy normal B-algebra will be anti fuzzy B-algebra(under s-norm S).

Proposition 4.3. If $\mu \in AFNBS(X)$. Then $\mu \in AFBS(X)$.

Proof. Let $x, y \in X$. Then

$$\mu(x * y) = \mu((x * y) * (0 * 0)) \leqslant S(\mu(x * 0), \mu(y * 0)) = S(\mu(x), \mu(y))$$

then $\mu \in AFBS(X)$. □

Remark 4.4. The converse of Proposition 4.3 is not true. For example, let $X = \{0, 1, 2, 3, 4, 5\}$ be a set given by the following Cayley table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define $\mu : X \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.1 & \text{if } x = 0, 3 \\ 0.7 & \text{if } x = 1, 2, 4, 5. \end{cases}$$

Let $S(a, b) = S_m(a, b) = \max\{a, b\}$ for all $a, b \in [0, 1]$ then $\mu \in AFBS(X)$. But since

$$\mu((2 * 5) * (4 * 1)) = \mu(2) = 0.7 \nleq S(\mu(2 * 4), \mu(5 * 1)) = S(\mu(2), \mu(2)) = S(0.1, 0.1) = 0.1$$

thus $\mu \notin AFNBS(X)$.

Proposition 4.5. Let $\mu \in AFNBS(X)$. Then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu(x * y) &= \mu((x * y) * (x * x)) \\ &\leqslant S(\mu(x * x), \mu(y * x)) \\ &= S(\mu(0), \mu(y * x)) \\ &= \mu(y * x) \\ &= \mu((y * x) * (y * y)) \\ &\leqslant S(\mu(y * y), \mu(x * y)) \\ &= S(\mu(0), \mu(x * y)) \\ &= \mu(x * y) \end{aligned}$$

then $\mu(x * y) = \mu(y * x)$. □

Proposition 4.6. Let $\mu \in AFNBS(X)$ and S be idempotent s-norm. Then

$$N = \{x \in X : \mu(x) = \mu(0)\}$$

will be a normal subalgebra of X.

Proof. Let $x, y, a, b \in X$. If $x * y \in N$ and $a * b \in N$, then $\mu(x * y) = \mu(a * b) = \mu(0)$. As $\mu \in AFNBS(X)$ so

$$\mu((x * a) * (y * b)) \leqslant S(\mu(x * y), \mu(a * b)) = S(\mu(0), \mu(0)) = \mu(0) \leqslant \mu((x * a) * (y * b))$$

thus $\mu((x * a) * (y * b)) = \mu(0)$ and then $(x * a) * (y * b) \in N$ thus N will be a normal subalgebra of X. □

Proposition 4.7. Let $\mu : X \to [0,1]$ be a fuzzy set and S be idempotent s-norm. If $\mu \in AFNBS(X)$, then the set

$$\mu_t = \{x \in X : \mu(x) \leqslant t\}$$

is either empty or normal subalgebra of B-algebra X for every $t \in [0,1]$.

Proof. Let $\mu \in AFNBS(X)$ and $\mu_t = \{x \in X : \mu(x) \leqslant t\}$ be not empty and $x * y, a * b \in \mu_t$. Then $\mu(x * y) \leqslant t$ and $\mu(a * b) \leqslant t$ such that

$$\mu((x * a) * (y * b)) \leqslant S(\mu(x * y), \mu(a * b)) \leqslant S(t,t) = t$$

which means that $(x * a) * (y * b) \in \mu_t$. Hence μ_t is normal subalgebra of B-algebra X for every $t \in [0,1]$, which proves the proposition. □

Proposition 4.8. Let $\mu \in AFNBS(X)$ and $\nu \in AFNBS(X)$. Then $\mu \cup \nu \in AFNBS(X)$.

Proof. Let $x, y, a, b \in X$. As

$$\begin{aligned}(\mu \cup \nu)((x * a) * (y * b)) &= S(\mu((x * a) * (y * b)), \nu((x * a) * (y * b))) \\ &\leqslant S(S(\mu(x * y), \mu(a * b)), S(\nu(x * y), \nu(a * b))) \\ &= S(S(\mu(x * y), \nu(x * y)), S(\nu(a * b), \nu(a * b))) \\ &= S((\mu \cup \nu)(x * y), (\mu \cup \nu)(a * b))\end{aligned}$$

so

$$(\mu \cup \nu)((x * a) * (y * b)) \leqslant S((\mu \cup \nu)(x * y), (\mu \cup \nu)(a * b)).$$

Then $\mu \cup \nu \in AFNBS(X)$. □

Proposition 4.9. Let $\mu \in AFNBS(X)$ and $\nu \in AFNBS(Y)$. Then $\mu \oplus \nu \in AFNBS(X \oplus Y)$.

Proof. Let $(x_1, x_2), (y_1, y_2), (a_1, a_2), (b_1, b_2) \in X \oplus Y$. Then

$$\begin{aligned}&(\mu \oplus \nu)(((x_1, x_2) * (a_1, a_2)) * ((y_1, y_2) * (b_1, b_2))) \\ &= (\mu \oplus \nu)((x_1 * a_1, x_2 * a_2) * (y_1 * b_1, y_2 * b_2)) \\ &= (\mu \oplus \nu)((x_1 * a_1) * (y_1 * b_1), (x_2 * a_2) * (y_2 * b_2)) \\ &= S(\mu((x_1 * a_1) * (y_1 * b_1)), \nu((x_2 * a_2) * (y_2 * b_2))) \\ &\leqslant S(S(\mu(x_1 * y_1), \mu(a_1 * b_1)), S(\nu(x_2 * y_2), \nu(a_2 * b_2))) \\ &= S(S(\mu(x_1 * y_1), \nu(x_2 * y_2)), S(\mu(a_1 * b_1), \nu(a_2 * b_2))) \\ &= S((\mu \oplus \nu)(x_1 * y_1, x_2 * y_2), (\mu \oplus \nu)(a_1 * b_1, a_2 * b_2)) \\ &= S((\mu \oplus \nu)((x_1, x_2) * (y_1, y_2)), (\mu \oplus \nu)((a_1, a_2) * (b_1, b_2)))\end{aligned}$$

and so

$$(\mu \oplus \nu)(((x_1, x_2) * (a_1, a_2)) * ((y_1, y_2) * (b_1, b_2))) \leqslant S((\mu \oplus \nu)((x_1, x_2) * (y_1, y_2)), (\mu \oplus \nu)((a_1, a_2) * (b_1, b_2))).$$

Thus $\mu \oplus \nu \in AFNBS(X \oplus Y)$. □

Proposition 4.10. Let $\mu_i \in AFNBS(X_i)$, where $1 \leqslant i \leqslant n$, then

$$\mu = \oplus_{i=1}^{n} \mu_i \in AFNBS(\oplus_{i=1}^{n} X_i = X)$$

such that

$$\mu(x) = (\oplus_{i=1}^{n} \mu_i)(x_1, x_2, ..., x_n) = S_n(\mu_1(x_1, \mu_2(x_2, ..., \mu_n(x_n))$$

for all $x = (x_1, x_2, ..., x_n) \in X$.

Proposition 4.11. If $\mu \in AFNBS(X)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be an epimorphic B-homomorphism of B-algebras, then $\varphi(\mu) \in AFNBS(Y)$.

Proof. Let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2, 3, 4$. Then

$$\begin{aligned}
\varphi(\mu)((y_1 \acute{*} y_2) \acute{*} (y_3 \acute{*} y_4)) &= \inf\{(\mu(x_1 * x_2) * (x_3 * x_4)) \mid x_i \in X, \varphi(x_i) = y_i\} \\
&\leqslant \inf\{S(\mu(x_1 * x_3), \mu(x_3 * x_4)) \mid x_i \in X, \varphi(x_i) = y_i\} \\
&= S(\inf\{\mu(x_1 * x_3) \mid x_i \in X, \varphi(x_i) = y_i\}, \inf\{\mu(x_2 * x_4) \mid x_i \in X, \varphi(x_i) = y_i\}) \\
&= S(\varphi(\mu)((y_1 \acute{*} y_3)), \varphi(\mu)((y_2 \acute{*} y_4)))
\end{aligned}$$

thus

$$\varphi(\mu)((y_1 \acute{*} y_2) \acute{*} (y_3 \acute{*} y_4)) \leqslant S(\varphi(\mu)((y_1 \acute{*} y_3)), \varphi(\mu)((y_2 \acute{*} y_4))).$$

Thus $\varphi(\mu) \in AFNBS(Y)$. □

Proposition 4.12. If $\nu \in AFNBS(Y)$ and $\varphi : (X; *, 0) \to (Y; \acute{*}, \acute{0})$ be a B-homomorphism of B-algebras, then $\varphi^{-1}(\nu) \in AFNBS(X)$.

Proof. Let $x_1, x_2, x_3, x_4 \in X$. Now

$$\begin{aligned}
\varphi^{-1}(\nu)((x_1 * x_2) * (x_3 * x_4)) &= \nu(\varphi((x_1 * x_2) * (x_3 * x_4)) \\
&= \nu(\varphi(x_1 * x_2) * \varphi(x_3 * x_4)) \\
&= \nu((\varphi(x_1) * \varphi(x_2)) * (\varphi(x_3) * \varphi(x_4))) \\
&\leqslant S(\nu(\varphi(x_1) * \varphi(x_3)), \nu(\varphi(x_2) * \varphi(x_4))) \\
&= S(\nu(\varphi(x_1 * x_3)), \nu(\varphi(x_2 * x_4))) \\
&= S(\varphi^{-1}(\nu)(x_1 * x_3), \varphi^{-1}(\nu)(x_2 * x_4))
\end{aligned}$$

then

$$\varphi^{-1}(\nu)((x_1 * x_2) * (x_3 * x_4)) \leqslant S(\varphi^{-1}(\nu)(x_1 * x_3), \varphi^{-1}(\nu)(x_2 * x_4)).$$

Therefore $\varphi^{-1}(\nu) \in AFNBS(X)$. □

5. Conclusion and open problem

In this paper, as using s-norms, we defined and introduced anti fuzzy B-subalgebras and anti fuzzy normal B-subalgebras and we investigated fundamental properties of them. Next, we investigated the union and direct sum of them and obtained characterizations of them by using B-homomorphisms of B-algebras. Now one can introduce anti fuzzy Q-subalgebras and anti fuzzy normal Q-subalgebras and obtain some results about them as we did for B-subalgebras and this can be an open problem.

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